

# 6.251/15.081J Recitation 6

## Week 6

### 1 Review Problems

**Example 1.1.** For each of the following claims, state whether true or false and justify briefly. Assume standard form unless otherwise stated.

- (a)  $\mathcal{N}(A)$  is a subspace.
  - (b) A rank-deficient matrix always has a nullspace of dimension greater than zero.
  - (c) The system  $A\mathbf{x} = \mathbf{0}$  has a nonzero solution if and only if the rows of  $A$  are linearly dependent.
  - (d) A nonempty polyhedron in standard form can contain at most  $\binom{n}{m}$  vertices, and there are LP instances for which this bound is tight.
  - (e) Degenerate BFSs result in a standard form system if and only if the system has redundant constraints.
  - (f) A BFS for a standard form system is degenerate if and only if multiple bases correspond to it.
  - (g) The polyhedron  $\{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} \leq \mathbf{0}\}$  has either zero or one extreme point.
  - (h) The tableau columns corresponding to slack variables are precisely the columns of  $B^{-1}$ .
  - (i) There exist multiple optimal solutions if and only if there exists a basis with some zero nonbasic reduced costs.
  - (j) Let  $(\mathbf{x}_B, \mathbf{x}_N)$  be a BFS and  $\bar{\mathbf{c}}$  be its reduced costs. For any other feasible solution  $\mathbf{y}$ , the cost difference between  $\mathbf{x}$  and  $\mathbf{y}$  is  $\bar{\mathbf{c}}_N^T \mathbf{y}_N$ .
-

- (k) Regardless of the pivoting rule employed, the simplex method will never cycle for a problem with one equality constraint.
- (l) An algorithm for finding a solution to a set of linear inequalities implies an algorithm for solving an LP, and vice versa.
- (m) The system  $A\mathbf{x} \leq \mathbf{b}$  is feasible if and only if the rows of  $A$  are linearly independent.
- (n) (Modified from [3], ch. 4). A system of linear inequalities is infeasible if and only if there exists a nonnegative linear combination of the inequalities which is inconsistent by itself.
- (o) Dual degeneracy implies primal non-uniqueness.
- (p) If the dual has multiple optimal solutions, then all optimal primal BFSs are degenerate.

**Example 1.2.** (Modified from [3], ch. 2).

Consider a standard form tableau at a non-optimal BFS. Suppose that for the problem we have the constraint  $\mathbf{1}^T \mathbf{x} \leq \alpha$ . Let  $\bar{c}_s = \min_{i=1, \dots, n} (\bar{c}_i)$  and  $z$  be the objective value of the current BFS. Prove that  $z + \alpha \bar{c}_s$  is a lower bound to the optimal value of the problem.

**Example 1.3.** (From [4], ch. 3).

Suppose an LP in standard form has two columns that are proportional with a positive constant of proportionality. Construct an equivalent LP with one fewer column.

**Example 1.4.** [1], exercise 4.19.

**Example 1.5.** (From [2], ch. 4).

*Interpretation of LP dual via relaxed problems.* Consider the inequality form LP

$$\begin{aligned} & \text{minimize} && \mathbf{c}^T \mathbf{x} \\ & \text{subject to} && A\mathbf{x} \leq \mathbf{b} \end{aligned}$$

with optimal value  $p^*$ . Suppose that  $\mathbf{w} \in \mathbb{R}^m$  satisfies  $\mathbf{w} \geq \mathbf{0}$ . If  $\mathbf{x}$  is feasible for the LP, it also satisfies the inequality

$$\mathbf{w}^T A\mathbf{x} \leq \mathbf{w}^T \mathbf{b}.$$

Geometrically, this means that for any  $\mathbf{w} \geq \mathbf{0}$ , the halfspace

$$H_w = \{ \mathbf{x} \in \mathbb{R}^n \mid (A^T \mathbf{w})^T \mathbf{x} \leq \mathbf{w}^T \mathbf{b} \}$$

contains the feasible set for the LP. Therefore if we minimize the objective  $\mathbf{c}^T \mathbf{x}$  over the halfspace  $H_w$  we get a lower bound on  $p^*$ .

- (a) Derive an expression for the minimum value of  $\mathbf{c}^T \mathbf{x}$  over the halfspace  $H_w$  as a function of  $\mathbf{w} \geq \mathbf{0}$ .
- (b) Now formulate the problem of finding the best such bound, by maximizing the bound obtained by choice of  $\mathbf{w} \geq \mathbf{0}$ .
- (c) Relate the result of (b) to the dual of the LP.

## References

- [1] Bertsimas, D.; Tsitsiklis, J.N. *Introduction to Linear Optimization*. Athena Scientific, 1997.
- [2] Boyd, S., Vandenberghe, L. *Convex Optimization*. Course reader. Stanford University, 2001.
- [3] Murty, K.G. *Linear and Combinatorial Programming*. John Wiley & Sons, 1976.
- [4] Papadimitriou, C.H.; Steiglitz, K. *Combinatorial Optimization: Algorithms and Complexity*. Prentice-Hall, 1982.