Dantzig–Wolfe decomposition

$$\begin{array}{rl} \text{minimize} & \mathbf{c}_1'\mathbf{x}_1 + \mathbf{c}_2'\mathbf{x}_2 \\ \text{subject to} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 = \mathbf{b}_0 \\ & \mathbf{F}_1\mathbf{x}_1 = \mathbf{b}_1 \\ & \mathbf{F}_2\mathbf{x}_2 = \mathbf{b}_2 \\ & \mathbf{x}_1, \mathbf{x}_2 \geq \mathbf{0} \end{array}$$

$$P_i = \{\mathbf{x}_i \geq \mathbf{0} \mid \mathbf{F}_i\mathbf{x}_i = \mathbf{b}_i\}$$

$$\begin{array}{rl} \text{minimize} & \mathbf{c}_1'\mathbf{x}_1 + \mathbf{c}_2'\mathbf{x}_2 \\ \text{subject to} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 = \mathbf{b}_0 \\ & \mathbf{x}_1 \in P_1 \\ & \mathbf{x}_2 \in P_2. \end{array}$$

- Assume P_1, P_2 bounded

- $\mathbf{x}_1^j$, $j \in J_1$, extreme points of P_1
- $\mathbf{x}_2^j$, $j \in J_2$, extreme points of P_2

$$\mathbf{x}_i = \sum_{j \in J_i} \lambda_i^j \mathbf{x}_i^j$$

where

$$\lambda_i^j \geq 0, \qquad \sum_{j \in J_i} \lambda_i^j = 1, \qquad i = 1, 2.$$

- Reformulated problem (master)

$$\begin{aligned} \text{minimize} \quad & \sum_{i=1,2} \sum_{j \in J_i} \lambda_i^j \mathbf{c}_i' \mathbf{x}_i^j \\ \text{subject to} \quad & \sum_{i=1,2} \sum_{j \in J_i} \lambda_i^j \mathbf{D}_i \mathbf{x}_i^j = \mathbf{b}_0 \\ & \sum_{j \in J_1} \lambda_1^j = 1 \\ & \sum_{j \in J_2} \lambda_2^j = 1 \\ & \lambda_i^j \geq 0, \qquad \forall i, j. \end{aligned}$$

$$\sum_{j \in J_1} \lambda_1^j \begin{bmatrix} \mathbf{D}_1 \mathbf{x}_1^j \\ 1 \\ 0 \end{bmatrix} + \sum_{j \in J_2} \lambda_2^j \begin{bmatrix} \mathbf{D}_2 \mathbf{x}_2^j \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \mathbf{b}_0 \\ 1 \\ 1 \end{bmatrix}.$$

- Fewer constraints, smaller tableau

- Many columns

Apply column generation ideas

- Given basis $\mathbf{B}$

- Assume $\mathbf{p}' = \mathbf{c}_B' \mathbf{B}^{-1}$ is available
  - Dimension $m_0 + 2$
  - $\mathbf{p} = (\mathbf{q}, r_1, r_2)$

Reduced cost of λ_1^j is

$$\mathbf{c}_1'\mathbf{x}_1^j - [\mathbf{q}' \; r_1 \; r_2] \begin{bmatrix} \mathbf{D}_1\mathbf{x}_1^j \\ 1 \\ 0 \end{bmatrix} = (\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1^j - r_1.$$

Use simplex to

$$\begin{aligned} \text{minimize} \quad & (\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1 \\ \text{subject to} \quad & \mathbf{x}_1 \in P_1, \end{aligned}$$

- If optimal cost $< r_1$:
  - optimal extreme point $\mathbf{x}_1^j$ with $(\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1^j < r_1$
  - λ_1^j has negative reduced cost
  - generate column and have it enter basis

$$\begin{bmatrix} \mathbf{D}_1\mathbf{x}_1^j \\ 1 \\ 0 \end{bmatrix}$$

- If optimal cost $\geq r_1$
  $(\mathbf{c}_1' - \mathbf{q}'\mathbf{D}_1)\mathbf{x}_1^j \geq r_1$ for all extreme points $\mathbf{x}_1^j$

- reduced cost of every λ_1^j is nonnegative.

Dantzig–Wolfe decomposition algorithm

- Start with $m_0 + 2$ extreme points of P_1 and P_2
- bfs of master problem
- dual vector $\mathbf{p}' = (\mathbf{q}, r_1, r_2)' = \mathbf{c}'_B \mathbf{B}^{-1}$.
- Form and solve the two subproblems
- If optimal costs $\geq r_1, r_2$, terminate
- If the optimal cost in the ith subproblem $< r_i$, some λ_i^j can become basic
- Do a revised simplex iteration

Why two subproblems?

$$\begin{array}{rl} \text{minimize} & \mathbf{c}'_1\mathbf{x}_1 + \mathbf{c}'_2\mathbf{x}_2 + \cdots + \mathbf{c}_t\mathbf{x}_t \\ \text{subject to} & \mathbf{D}_1\mathbf{x}_1 + \mathbf{D}_2\mathbf{x}_2 + \cdots + \mathbf{D}_t\mathbf{x}_t = \mathbf{b}_0 \\ & \mathbf{F}_i\mathbf{x}_i = \mathbf{b}_i, \qquad i = 1, 2, \ldots, t, \\ & \mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t \geq \mathbf{0}. \end{array}$$

Can have $t = 1$

$$\begin{array}{rl} \text{minimize} & \mathbf{c}'\mathbf{x} \\ \text{subject to} & \mathbf{D}\mathbf{x} = \mathbf{b}_0 \\ & \mathbf{F}\mathbf{x} = \mathbf{b} \\ & \mathbf{x} \geq \mathbf{0}, \end{array}$$

$P = \{\mathbf{x} \geq \mathbf{0} \mid \mathbf{F}\mathbf{x} = \mathbf{b}\}$

Example

$$\begin{array}{rl} \text{minimize} & -4x_1 - x_2 - 6x_3 \\ \text{subject to} & 3x_1 + 2x_2 + 4x_3 = 17 \\ & 1 \le x_1 \le 2 \\ & 1 \le x_2 \le 2 \\ & 1 \le x_3 \le 2 \end{array}$$

$\mathbf{D} = [3\ 2\ 4]$, $\mathbf{b}_0 = 17$
$\mathbf{x} \in P = \{\mathbf{x} \in \Re^3 \mid 1 \le x_i \le 2,\ i = 1, 2, 3\}$
Master problem

$$\sum_{j=1}^{3} \lambda_j \mathbf{D}\mathbf{x}^j = 17,$$

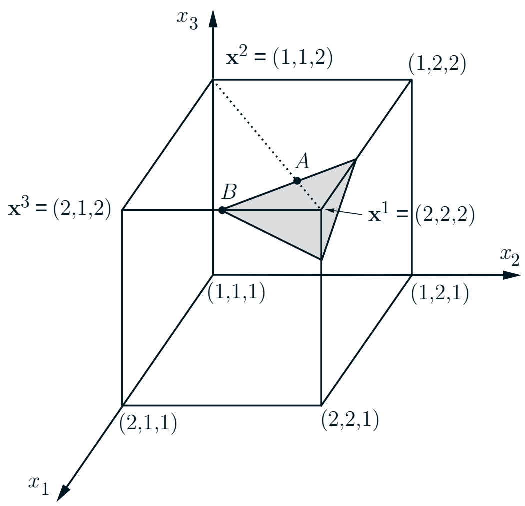
$$\sum_{j=1}^{3} \lambda_j = 1,$$

Columns $(\mathbf{D}\mathbf{x}^j, 1)$.

Let $\mathbf{x}^1 = (2, 2, 2)$, $\mathbf{x}^2 = (1, 1, 2)$
Let λ_1, λ_2 initial basic variables.

$[\mathbf{q}'\ r] = \mathbf{p}' = \mathbf{c}_B'\mathbf{B}^{-1} = [-22\ -17]\mathbf{B}^{-1} = [-1\ -4]$.

- minimize $(\mathbf{c}' - \mathbf{q}'\mathbf{D})\mathbf{x}$ subject to $\mathbf{x} \in P$.
- $\mathbf{c}' - \mathbf{q}'\mathbf{D} = (-1, 1, -2)'$
- optimal solution $\mathbf{x}^3 = (2, 1, 2)$, cost -5
- Let λ_3 enter the basis
- Resulting bfs is optimal



Starting the algorithm

- Find extreme points $\mathbf{x}_1^1 \in P_1$ and $\mathbf{x}_2^1 \in P_2$

- Assume $\mathbf{D}_1\mathbf{x}_1^1 + \mathbf{D}_2\mathbf{x}_2^1 \leq \mathbf{b}$

- Auxiliary problem

$$\begin{aligned} \text{minimize} \quad & \sum_{t=1}^{m_0} y_t \\ \text{subject to} \quad & \sum_{i=1,2} \sum_{j \in J_i} \lambda_i^j \mathbf{D}_i \mathbf{x}_i^j + \mathbf{y} = \mathbf{b}_0 \\ & \sum_{j \in J_1} \lambda_1^j = 1 \\ & \sum_{j \in J_2} \lambda_2^j = 1 \\ & \lambda_i^k \geq 0, \quad y_t \geq 0 \qquad \forall i, j, t. \end{aligned}$$

- Initial bfs: $\lambda_1^1 = \lambda_2^1 = 1$
  $\mathbf{y} = \mathbf{b}_0 - \mathbf{D}_1\mathbf{x}_1^1 - \mathbf{D}_2\mathbf{x}_2^1$.

- Revised simplex method:
  - Terminates for nondegenerate problems
  - Can use lexicographic pivoting rule
- In practice
  - Fast progress initially
  - Slow progress towards the end
  - May wish to terminate prematurely
- Storage $O((m_0 + t)^2) + t \cdot O(m_1^2)$
- Compare to $O((m_0 + tm_1)^2)$

Bounds on optimal cost

- z^*: optimal cost in master problem
- z cost of current feasible solution
- r_i value of dual variable
- z_i optimal cost in ith subproblem
- Then,

$$z + \sum_i (z_i - r_i) \le z^* \le z.$$

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Proof

- $\mathbf{x}$ basic feasible solution

- $\mathbf{y}$ other feasible solution

- $\mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}\mathbf{y} = \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{b} = \mathbf{c}'\mathbf{x}$
  $\mathbf{c}'\mathbf{y} = \overline{\mathbf{c}}'\mathbf{y} + \mathbf{c}'_B\mathbf{B}^{-1}\mathbf{A}\mathbf{y} = \overline{\mathbf{c}}'\mathbf{y} + \mathbf{c}'\mathbf{x}$.

- $\mathbf{y}$ feasible solution of master problem

- $\mathbf{x}$ current bfs of master problem
  $\mathbf{c}'\mathbf{x} = z$

- Reduced cost of $\lambda_i^j \geq z_i - r_i$

  $$\mathbf{c}'\mathbf{y} \geq \sum_i \sum_{j \in J_i} \lambda_i^j (z_i - r_i) + z = \sum_i (z_i - r_i) + z.$$