MASSACHUSETTS INSTITUTE OF TECHNOLOGY

Fall 2002 — 6.251/15.081

Quiz 1 — 10/16/02 (7:30-9:30 pm)

Problem 1. (30 points)

For each one of the statements below, state whether it is true or false. Include a 1-3 line supporting sentence or drawing, enough to convince us that you are not guessing the answer, but not a comprehensive, rigorous formal justification.

(a) There exist nonempty and bounded polyhedra of the form $\{\mathbf{x} \in \Re^n \mid \mathbf{Ax} \leq \mathbf{b}\}$, in which every basic solution is also a vertex.

(b) Consider the problem of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{Ax} \leq \mathbf{b}$. If we increase some component of $\mathbf{b}$, then the optimal cost cannot increase.

(c) A feasible linear program of minimizing $\mathbf{c}'\mathbf{x}$ subject to $\mathbf{Ax} \geq \mathbf{b}$ has finite cost if and only if $\mathbf{c}$ is a nonnegative combination of the rows of $\mathbf{A}$.

For the next three statements, assume that we are dealing with a linear programming (minimization) problem in standard form, with the matrix $\mathbf{A}$ having full row rank.

(d) If a tie occurs while choosing the pivot row during the primal simplex method, then the next basic feasible solution will be degenerate.

(e) If the optimal cost is $-\infty$, then the right hand side vector $\mathbf{b}$ can be adjusted in some way to make the optimal cost finite.

(f) The dual of the Phase I problem will never have an infinite optimal cost.

Problem 2. (24 points)

We are given a standard form problem, with $\mathbf{A}$ having full row rank. For each of the following three tasks, describe an algorithmic method that accomplishes the task. Your answer can be of the form: "solve such and such an LP and decide 'yes' if the outcome of the LP is such and such."

(a) Given also an optimal solution $\mathbf{x}^*$ and a corresponding final full tableau, decide whether $\mathbf{x}^*$ is the unique optimal solution.

(b) Decide whether the feasible set is unbounded.

(c) Decide whether the first component x_1 is equal to zero in every optimal solution. (For full credit, do that by solving a single LP, and without assuming that an optimal solution or the optimal cost is given to you.)

Problem 3. (26 points)

Consider the problem of minimizing $\max\{\mathbf{c}'\mathbf{x}, \mathbf{d}'\mathbf{x}\}$, subject to $\mathbf{x} \in P$, where $P = \{\mathbf{x} \mid \mathbf{A}\mathbf{x} \leq \mathbf{b}\}$.

(a) Formulate this problem as a linear programming problem.

(b) Write down the dual problem.

(c) Assume that there is an optimal solution, and also that P has at least one extreme point. Show that there is an optimal solution which is either an extreme point of P, or lies on a edge of P. (Hint: An "edge" can be defined in terms of the number of linearly independent active constraints.)

Problem 4. (20 points)

Suppose that the zero vector in $\Re^n$ cannot be expressed as a convex combination of some given vectors $\mathbf{x}^1, \ldots, \mathbf{x}^K \in \Re^n$. Show that there exists some $\mathbf{c} \in \Re^n$ such that $\mathbf{c}'\mathbf{x}^i > 0$ for every $i \in \{1, \ldots, K\}$.

Note: Derive this result without explicitly invoking the separating hyperplane theorem or results on representation of polyhedra presented in last week's lecture.