MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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Quiz 3

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1. (25 points)
Consider an undirected graph $G = (\mathcal{N}, \mathcal{E})$ with n nodes. A *clique* is a fully connected set of nodes. That is, a clique consists of a set of nodes S such that if $i, j \in S$, then $\{i, j\} \in \mathcal{E}$. Formulate the problem of finding a maximum size clique (i.e., with the largest possible number of nodes) as an integer programming problem.

Note: There is a trivial formulation with exponentially many variables. We are looking for an economical one, with polynomially many variables.

2. (25 points) Consider a directed graph, and let us fix an origin node s and a destination node t. We define the *connectivity* of the graph as the maximum number of directed paths from s to t that do not share any nodes. We define the *vulnerability* of the graph as the minimum number of nodes (besides s and t) that need to be removed so that there exists no directed path from s to t. Prove that connectivity is equal to vulnerability.
Note: Assume for simplicity that there are no arcs incoming to s or outgoing from t.

3. (25 points)
Consider an (all-to-one) shortest path problem together with the standard formulation as a network flow problem. Suppose that there exists a negative cost directed cycle.

(a) What can you say about the dual of the network flow problem? Is it feasible? Does it have one, many, or no optimal solutions?

(b) What can you say about the solutions to Bellman's equation? (Does it have many, one, or none?)
Provide a brief justification for your answers.

4. (25 points)
Consider the following variation of the zero-one Knapsack problem:

$$\begin{aligned} \text{minimize} \quad & \sum_i c_i x_i \\ \text{subject to} \quad & \sum_i w_i x_i \geq W \\ & x_i \in \{0,1\} \end{aligned}$$

We will use Lagrangian relaxation, and associate a Lagrange multiplier to the inequality constraint $\sum_i w_i x_i \geq W$.

(a) Write down an expression for the dual function $Z(p)$. (You do not have to simplify this expression or carry out the minimizations.)

(b) Let $Z_D = \max_{p \geq 0} Z(p)$. What is the most that can you say about the relation between Z_D and Z_{LP}? (One-line answer.)

(c) If the Knapsack problem is to be solved using a branch and bound approach, can this Lagrangian relaxation be of use? (Very brief answer.)

Solutions

1.

maximize $y + 1 + \cdots + y_n$
subject to $x_{\{i,j\}} \geq y_i + y_j - 1$
$x_{\{i,j\}}, y_i \in \{0, 1\}$

2.

Let us remove all the arcs coming into s and all the arcs outgoing from t. Since we are interested in the paths from s to t, this is without loss of generality.

We construct a new graph as follows. For each node i different from s and t, we introduce instead two nodes i' and i'', connected by a unit capacity arc (i', i''). All the arcs (k, i) in the old graph are replaced by arcs (k, i'), and all the arcs (i, j) are replaced by arcs (i'', j). We let the capacities of these latter arcs be infinite. (Note that this is the same as the transformation introduced in Figure 7.8 of the text to model node capacities.) Any (integer) flow in this new graph consists of a set of node disjoint paths in the original graph. We conclude that connectivity equals maximum flow value in the auxiliary problem.

Consider a minimum cut in the auxiliary problem. Since its capacity is finite, the only arcs that cross the cut are arcs of the form (i', i''). Removing all such arcs that cross the cut is equivalent to removing the corresponding nodes i in the original network, thereby disconnecting s from t. Furthermore, there is a one-to-one correspondence between cuts and choices of nodes whose removal disconnects s from t in the original graph. This shows that the minimum cut capacity is equal to vulnerability. The desired result follows from the max-flow min-cut theorem.

3.

(a) The optimal primal cost is $-\infty$, which implies that the dual problem must be infeasible.

(b) Let $i_1, i_2, \ldots, i_n, i_1$ be a negative cost cycle. If Bellman's equation had a solution, it would satisfy $p_{i_1} \leq p_{i_2} + c_{i_1 i_2}, \ldots p_{i_n} \leq p_{i_1} + c_{i_n i_1}$. By adding these inequalities, we obtain $0 \leq c_{i_1 i_2} + \cdots + c_{i_n i_1} < 0$. The contradiction implies that there is no negative cost cycle.

4.

(a) $Z(p) = \min_{x_i \in \{0,1\}} \left[\sum_i c_i x_i + p\left(W - \sum_i w_i x_i\right)\right]$

(b) $Z_D = Z_{LP}$

(c) No. Lagrangian relaxation would be just a time consuming way of computing the readily computable quantity Z_{LP}.