6.251/15.081J Quiz 1 Solutions

Massachusetts Institute of Technology

Week 8

Problem 1.

(a) **True**. Consider $P = \{x \in \mathbb{R}^1 \mid -x \leq 0,\ x \leq 1\}$. Then $x = 0$ and $x = 1$ are the only basic solutions, but they are also vertices of P.

(b) **True**. Increasing a component of $\boldsymbol{b}$ gives a feasible set which is no smaller, so the optimal cost cannot increase. (A more complicated way to see this is to take the dual and invoke a shadow price argument using the sign of the dual variables).

(c) **True**. The dual problem is written as

$$\begin{array}{ll} \text{maximize} & \boldsymbol{b}^T \boldsymbol{p} \\ \text{subject to} & A^T \boldsymbol{p} = \boldsymbol{c} \\ & \boldsymbol{p} \geq 0. \end{array}$$

If $\boldsymbol{c}$ is a nonnegative combination of the rows of A, then the dual is feasible. Since we are given that the primal is feasible, it follows that both problems have equal and bounded optimal costs. Conversely, if the primal has a bounded optimal cost, the dual must also. In particular, the dual must be feasible, which implies that $\boldsymbol{c}$ is a nonnegative combination of the rows of A.

(d) **True**. Let $\theta^\star$ be the value of the min-ratio test. Then all basic variables x_j which have this value of $\theta_j = x_j/u_j$ where u_j is the jth component of the basic direction become $x_j - \theta^\star u_j = x_j - \theta_j u_j = 0$. Since there are more than one of these such variables, at least one stays in the basis, which implies that the resulting BFS is degenerate.

(e) **False**. If an LP is unbounded, its dual must be infeasible. But dual feasibility depends in no way on the vector $\boldsymbol{b}$.

(f) **True**. If the dual of the phase I formulation has an infinite optimal cost, then the phase I problem must be infeasible. This cannot be true, since $(\boldsymbol{x}, \boldsymbol{y}) = (\boldsymbol{0}, \boldsymbol{b})$ is always used as a starting BFS for the phase I problem.

Problem 2.

(a) Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & \mathbf{1}^T \boldsymbol{d}_N \\ \text{subject to} \quad & \bar{\boldsymbol{c}}_N^T \boldsymbol{d}_N = 0 \\ & B^{-1} A_N \boldsymbol{d}_N \leq B^{-1} \boldsymbol{b} \\ & \boldsymbol{d}_N \geq \mathbf{0}, \end{aligned}$$

where $\bar{\boldsymbol{c}}_N$ are the nonbasic reduced costs, $B^{-1}A_N$ are the nonbasic columns of the tableau, and $B^{-1}\boldsymbol{b}$ are the values of the basic variables. Assume without loss of generality that the basic variables are in order as the first m variables. If this LP has a nonzero solution $\boldsymbol{d}_N^\star$, then we can add $(-B^{-1}A_N\boldsymbol{d}_N^\star, \boldsymbol{d}_N^\star)$ to our current BFS and reach another feasible solution with the same cost. Conversely, if the optimal value of this LP is zero, then there is no basic direction in which we can travel and maintain the same cost, so our current solution is unique. Thus, there are multiple optimal solutions if and only if the LP above has a nonzero solution.

(b) Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & \mathbf{1}^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0. \end{aligned}$$

If the feasible set is unbounded, it must be that some variables can be extended to $+\infty$ because of the nonnegativity constraints. So the LP above will also be unbounded. Conversely, if the feasible set is bounded, then the LP above will have a finite optimal value. So the feasible set is unbounded if and only if the constructed LP is unbounded.

(c) Consider the following LP:

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{e}_1^T \boldsymbol{x} \\ \text{subject to} \quad & A\boldsymbol{x} = \boldsymbol{b} \\ & \boldsymbol{x} \geq 0 \\ & A^T\boldsymbol{p} \leq \boldsymbol{c} \\ & \boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{p}. \end{aligned}$$

If the problem is feasible, then any solution $(\boldsymbol{x}, \boldsymbol{p})$ is primal and dual feasible in the original problem (and hence $\boldsymbol{x}$ is an optimal solution to the original problem). Furthermore, if the optimal value of the above LP is zero, then it follows that no optimal solutions to the original problem have $x_1 > 0$. If there exist solutions with $\boldsymbol{e}_1^T\boldsymbol{x}$ greater than zero, then this is an optimal solution to the original problem with $x_1 > 0$.

If the constructed LP is infeasible, then the original problem is either infeasible or unbounded. In either case, there are no optimal solutions, so there are certainly no optimal solutions with $x_1 > 0$.

Problem 3.

(a) The formulation is

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & A\boldsymbol{x} \leq \boldsymbol{b} \\ & \mathbf{c}^T\boldsymbol{x} \leq t \\ & \boldsymbol{d}^T\boldsymbol{x} \leq t. \end{aligned}$$

(b) With m as the number of rows of A, $\boldsymbol{p} \in \mathbb{R}^m$, and $q, r \in \mathbb{R}$, we have the dual as:

$$\begin{aligned} \text{maximize} \quad & \boldsymbol{b}^T\boldsymbol{p} \\ \text{subject to} \quad & A^T\boldsymbol{p} + q\mathbf{c} + r\boldsymbol{d} = \mathbf{0} \\ & q + r = -1 \\ & \boldsymbol{p} \leq \mathbf{0} \\ & q, r \leq 0. \end{aligned}$$

(c) Let

$$Q = \{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} \mid A\boldsymbol{x} \leq \boldsymbol{b},\ \mathbf{c}^T\boldsymbol{x} - t \leq 0,\ \boldsymbol{d}^T\boldsymbol{x} - t \leq 0\}.$$

We now distinguish two cases.

Case I. Q has an extreme point. Then the LP over Q has an extreme point which is optimal, which will be a point with $n+1$ linearly independent active constraints. At least $n-1$ of these must come from the constraints $A\boldsymbol{x} \leq \boldsymbol{b}$, so the resulting $\boldsymbol{x}$ will at the very least be on an edge of P. If in fact there are n linearly independent active constraints from the first set, then we are at an extreme point of P.

Case II. Q has no extreme points. In this case, the claim is actually false. So, we must assume that Q has extreme points, or make another assumption that will get us to the desired result.

The simplest assumption that will do is that P has an extreme point, i.e., that the rows of A span $\mathbb{R}^n$. In that case, the matrix in part **(a)** is

$$\begin{bmatrix} A & \mathbf{0} \\ \mathbf{c}^T & -1 \\ \boldsymbol{d}^T & -1 \end{bmatrix}$$

The last row is independent from the rows of $[A\ \mathbf{0}]$, and therefore, there are $n+1$ linearly independent rows so Q has an extreme point.

Note: Only *Case I* was required for full credit.

Problem 4.

Consider the LP:

$$\begin{array}{ll} \text{maximize} & \mathbf{0}^T \boldsymbol{p} \\ \text{subject to} & X\boldsymbol{p} = \mathbf{0} \\ & \mathbf{1}^T \boldsymbol{p} = 1 \\ & \boldsymbol{p} \geq \mathbf{0} \end{array}$$

where X is the matrix with columns $\boldsymbol{x}^1, \ldots, \boldsymbol{x}^K$. By the statement in the problem, this LP is infeasible. Now consider the primal problem which has the above problem as its dual:

$$\begin{array}{ll} \text{minimize} & z \\ \text{subject to} & \boldsymbol{y}^T \boldsymbol{x}^i + z \geq 0,\ \ i = 1, \ldots, K. \end{array}$$

Since $\boldsymbol{y} = \mathbf{0}$, $z = 0$ is a feasible solution to the primal problem and its dual is infeasible, the primal must be unbounded. Thus, there exists a feasible $(\hat{\boldsymbol{y}}, \hat{z})$ with $\hat{z} < 0$ satisfying $\hat{\boldsymbol{y}}^T \boldsymbol{x}^i \geq -\hat{z},\ \ i = 1, \ldots, K$. Taking $\boldsymbol{c} = \hat{\boldsymbol{y}}$ proves the claim.