Lecture 3 - Axioms of Consumer Preference and the Theory of Choice

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Agenda:

1. Consumer preference theory
   (a) Notion of utility function
   (b) Axioms of consumer preference
   (c) Monotone transformations
2. Theory of choice
   (a) Solving the consumer's problem
      - Ingredients
      - Characteristics of the solution
      - Interior vs corner solutions
   (b) Constrained maximization for consumer
   (c) Interpretation of the Lagrange multiplier

Road map:
Theory

1. Consumer preference theory
2. Theory of choice
3. Individual demand functions
4. Market demand

Applications

1. Dead weight loss of Christmas
2. Food stamps and other taxes and transfers
3. Giffen goods: Theory and evidence

1 Consumer Preference Theory

A consumer's utility from consumption of a given bundle "A" is determined by a personal utility function.

1.1 Cardinal and ordinal utility

- Cardinal Utility Function

  According to this approach $U(A)$ is a cardinal number, that is:

  U : *consumption bundle* $\longrightarrow$ R^1 measured in "utils"

- Ordinal Utility Function

  More general than cardinal utility function

  U provides a "ranking" or "preference ordering" over bundles.

$$U : (A, B) \longrightarrow \begin{cases} A \; ^P \; B \\ B \; ^P \; A \\ A \; ^I \; B \end{cases}$$

  Used in demand/consumer theory

- Cardinal vs Ordinal Utility Functions

The problem with cardinal utility functions comes from the difficulty in finding the appropriate measurement index (metric).

Example: Is 1 util for person 1 equivalent to 1 util for person 2?

Or if we increase a person's utility from 1 to 2, is she twice as happy?

By being unit-free ordinal utility functions avoid these problems.

What's important about utility functions is that it allows us to model how people make personal choices, that is, how they choose among competing alternatives. We do not 't need to know how many "utils" people experience from each choice to answer this question; we just need to know how they rank choices.

Note: It's much harder to model interpersonal comparisons of utility

1.2 Axioms of Consumer Preference Theory

Created for purposes of:

1. Using mathematical representation of utility functions
2. Portraying rational behavior (rational in this case means 'optimizing')
3. Deriving "well-behaved" demand curves

1.2.1 Axiom 1: Preferences are Complete ("Completeness")

For any two bundles A and B, a consumer can establish a preference ordering. That is, for any comparison of bundles, she will choose one and only one of the following:

1. $A\ ^{P}\ B$
2. $B\ ^{P}\ A$
3. $A\ ^{I}\ B$

Without this property, preferences are undefined.

1.2.2 Axiom 2: Preferences are Transitive ("Transitivity")

For any consumer if $A\ ^{P}\ B$ and $B\ ^{P}\ C$ then it must be that $A\ ^{P}\ C$.

Consumers are consistent in their preferences.

1.2.3 Axiom 3: Preferences are Continuous ("Continuity")

If $A\ ^{P}\ B$ and C lies within an ε radius of B then $A\ ^{P}\ C$.

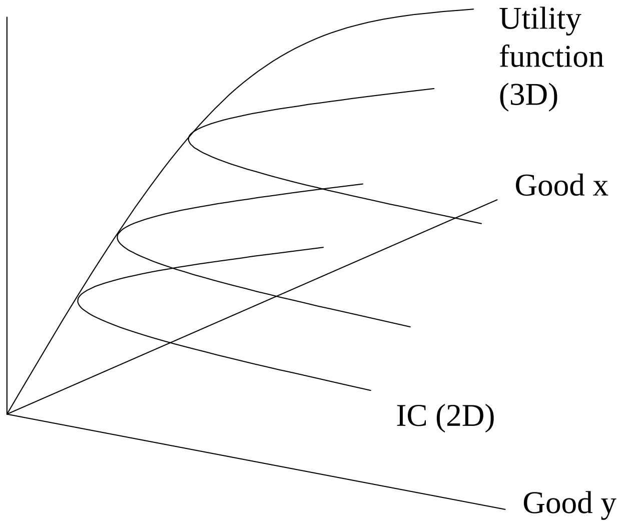
We need continuity to derive well-behaved demand curves.

Given Axioms 1- 3 are obeyed we can always define a utility function.

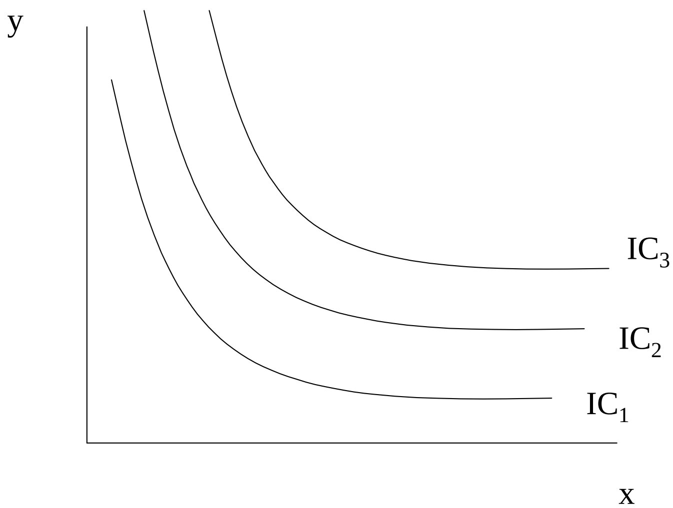
Any utility function that satisfies Axioms 1- 3 cannot have indifference curves that cross.

Indifference Curves Define a level of utility say $U(x) = \overline{U}$ then the indifference curve for $\overline{U}$, $IC(\overline{U})$ is the locus of consumption bundles that generate utility level $\overline{U}$ for utility function $U(x)$.

An Indifference Curve Map is a sequence of indifference curves defined over every possible bundle and every utility level: $\{IC(0), IC(\varepsilon), IC(2\varepsilon), ...\}$ with $\varepsilon = epsilon$



Indifference curves are level sets of this utility function.



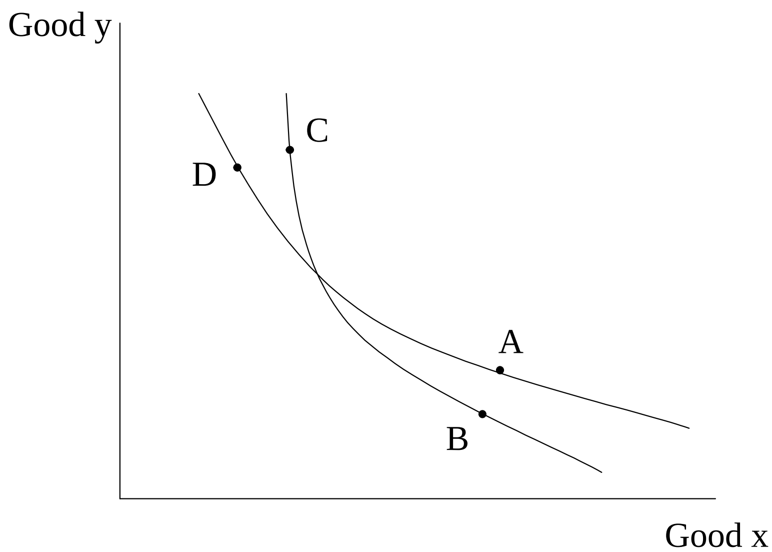
$$\left.\begin{array}{l} IC_3 \longrightarrow \text{ Utility level } U_3 \\ IC_2 \longrightarrow \text{ Utility level } U_2 \\ IC_1 \longrightarrow \text{ Utility level } U_1 \end{array}\right\} U_3 > U_2 > U_1$$

This is called an Indifference Curve Map

Properties:

- Every consumption bundle lies on some indifference curve (by the completeness axiom)
- INDIFFERENCE CURVES CANNOT INTERSECT (by the transitivity axiom)

Proof: say two indifference curves intersect:



According to these indifference curves:

$A \ ^P \ B$

$B \ ^I \ C$

$C \ ^P \ D$

$D \ ^I \ A$

By the above mentioned axioms: $A \ ^P \ D$ and $A \ ^I \ D$

which is a contradiction.

Axioms 4. and 5. are introduced to reflect observed behavior. They simplify problems greatly, but they are not necessary for a theory of rational choice.

1.2.4 Axiom 4: Non-Satiation (Never Get Enough)

Given two bundles, A and B, composed of two goods, X and Y.

X_A = amount of X in A, similarly X_B

Y_A = amount of Y in A, similarly Y_B

If $X_A = X_B$ and $Y_A > Y_B$ (assuming utility is increasing in both arguments) then $A \ ^P \ B$ (regardless of the levels of X_A, X_B, Y_A, Y_B)

This implies that:

1. The consumer always places positive value on more consumption
2. Indifference curve map stretches out endlessly (there is no upper limit to utility)

1.2.5 Axiom 5: Diminishing Marginal Rate of Substitution

In order to define this axiom we need to introduce the concept of Marginal Rate of Substitution and some further preliminary explanations.

Definition: MRS measures willingness to trade one bundle for another.

Example:

Bundle A = (6 hours of sleep, 50 points on the problem set)

Bundle B = (5 hours of sleep, 60 points on the problem set)

A and B lie on the same indifference curve

A student is willing to give up 1 more hour of sleep for 10 more points on the problem set.

Her willingness to substitute sleep for grade points at the margin (i.e. for 1 fewer hours of sleep) is: $\frac{10}{-1} = -10$

$$MRS\ (sleep\ for\ points) = |-10| = 10$$

MRS is measured along an indifference curve and it may vary along the same indifference curve. If so, we must define the MRS relative to some bundle (starting point).

$dU = 0$ along an indifference curve

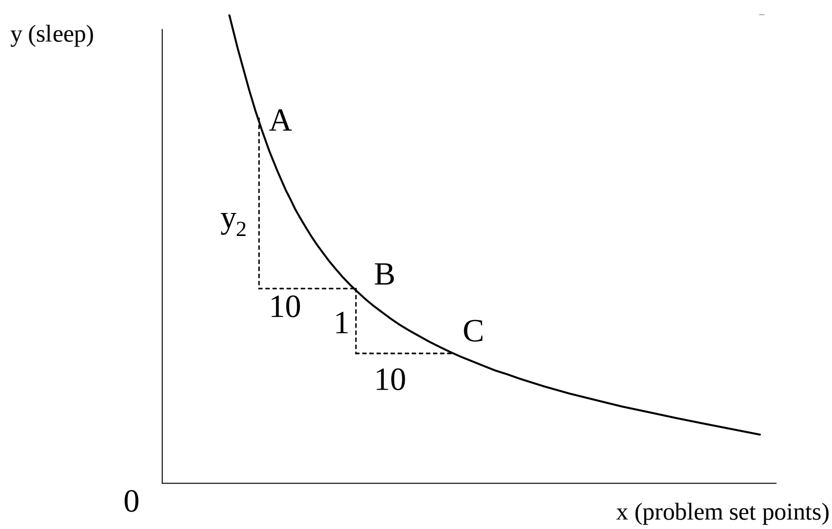
Therefore:

$$\begin{aligned} 0 &= \frac{\partial U}{\partial x}dx + \frac{\partial U}{\partial y}dy \\ 0 &= MU_x dx + MU_y dy \\ -\frac{dy}{dx} &= \frac{MU_x}{MU_y} = MRS\ of\ x\ for\ y \end{aligned}$$

MRS must always be evaluated at some particular point (consumption bundle) on the indifference curve.

So one should really write MRS$(\overline{x}, \overline{y})$ where$(\overline{x}, \overline{y})$ is a particular consumption bundle.

We are ready to explain what is meant by Diminishing Marginal Rate of Substitution.



MRS of x for y decreases as we go down on the indifference curve.

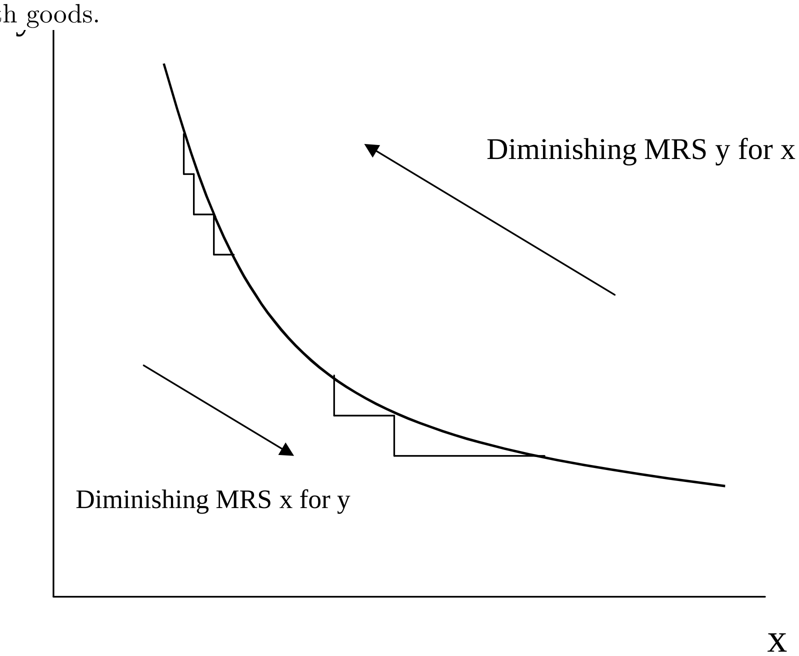
This indifference curve exhibits diminishing MRS: the rate at which (at the margin) a consumer is willing to trade x for y diminishes as the level of x consumed goes up.

That is the slope of the indifference curve between points B and C is less than the slope of the curve between points A and B.

Diminishing MRS is a consequence of Diminishing Marginal Utility.

A utility function exhibits diminishing marginal utility for good x when MU_x decreases as consumption of x increases.

A bow-shaped-to-origin (convex) indifference curve is one in which utility function has diminishing MU for both goods.



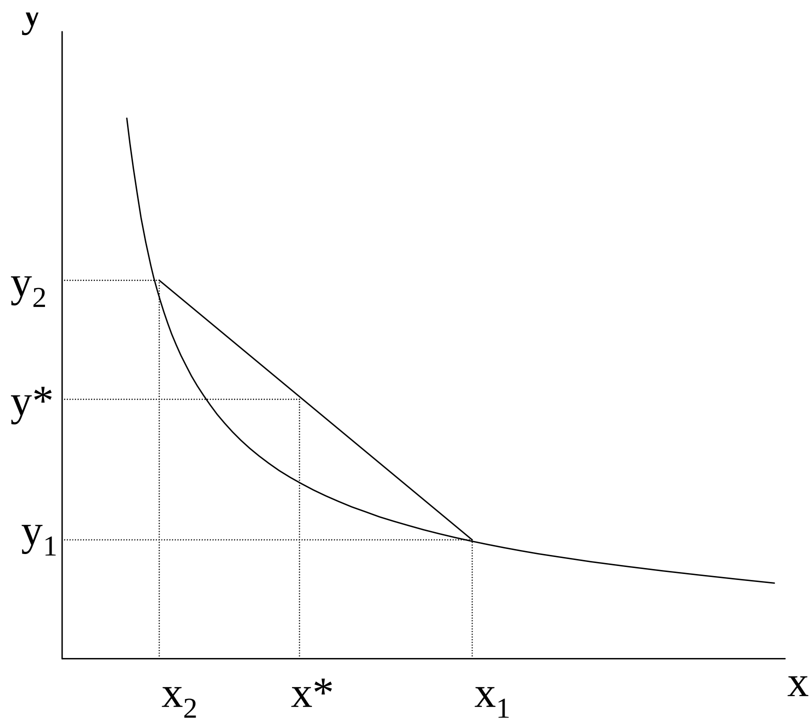
This implies that consumer prefers diversity in consumption.

An alternative definition of diminishing MRS can be given through the mathematical notion of *convexity*.

Definition: a function $U(x, y)$ is convex if:

$$U(\alpha x_1 + (1-\alpha)x_2, \alpha y_1 + (1-\alpha)y_2) \geq \alpha U(x_1, y_1) + (1-\alpha)U(x_2, y_2)$$

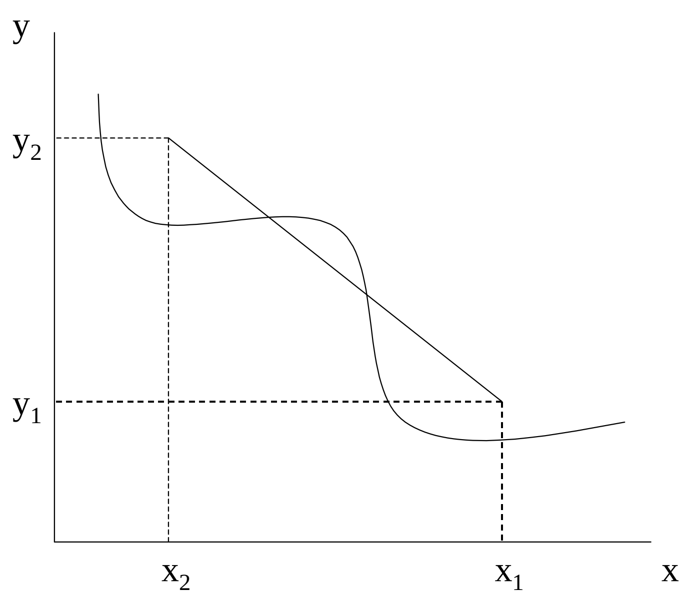
Suppose the two bundles, (x_1, y_1) and (x_2, y_2) are on the same indifference curve. This property states that the convex combination of this two bundles is on higher indifference curve than the two initial ones.



where $x^* = \alpha x_1 + (1-\alpha)x_2$ and $y^* = \alpha y_1 + (1-\alpha)y_2$.

This is verified for every $\alpha \in (0,1)$.

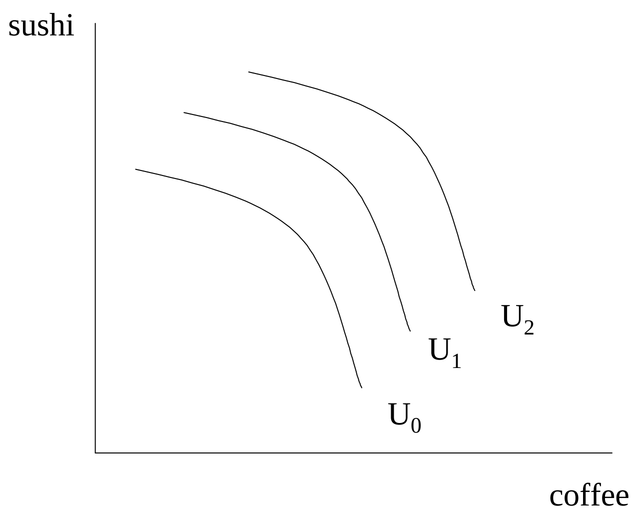
The following is an example of a non-convex curve:



In this graph not every point on the line connecting two points above the curve is also above the curve, therefore the curve is not convex.

Q: Suppose coffee and sushi have the same quality: the more you consume, the more you want. How do we draw this? For a given budget, should you diversify if you have this kind of preferences?

No, because preferences are not convex.



1.3 Cardinal vs Ordinal Utility

A utility function of the form $U(x,y) = f(x,y)$ is cardinal in the sense that it reads off "utils" as a function of consumption.

Obviously we don't know what utils are or how to measure them. Nor do we assume that 10 utils is twice as good as 5 utils. That is a cardinal assumption.

What we really care about is the ranking (or ordering) that a utility function gives over bundles of goods. Therefore we prefer to use ordinal utility functions.

We want to know if $A \, ^{P} \, B$ but not by how much.

However we do care that the MRS along an indifference curve is well defined , i.e. we do want to know precisely how people trade off among goods in indifferent (equally preferred) bundles.

Q: How can we preserve properties of utility that we care about and believe in (1.ordering is unique and 2. MRS exists) without imposing cardinal properties?

A: We state that utility functions are only defined up to a "*monotonic transformation*".

Definition: Monotonic Transformation

Let I be an interval on the real line (R^1) then: $g : I \longrightarrow R^1$ is a monotonic transformation if g is a strictly increasing function on I.

If $g(x)$ is differentiable then $g'(x) > 0 \; \forall x$

Informally: A monotone transformation of a variable is a *rank-preserving* transformation. [Note: not all rank-preserving transformations are differentiable.]

Examples: Which are monotone functions

Let y be defined on R^1:

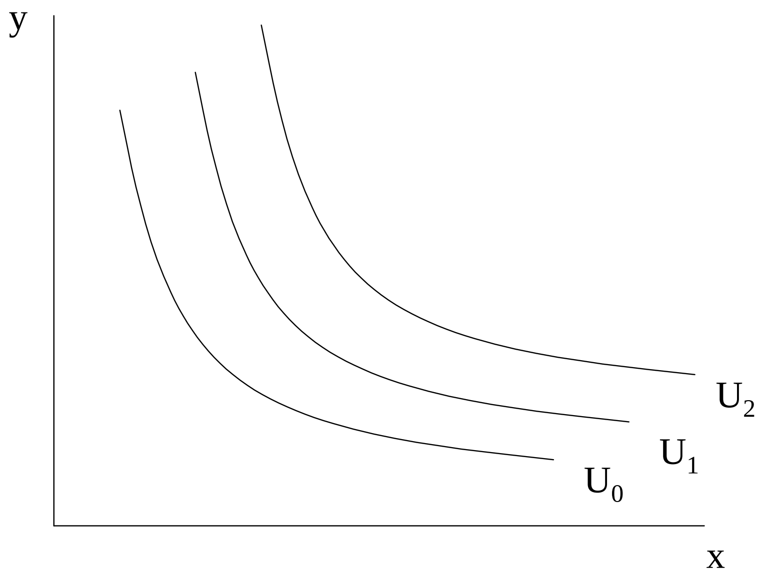
1. $x = y + 1$ [yes]
2. $x = 2y$ [yes]
3. $x = \exp(y)$ [yes]
4. $x = abs(y)$ [no]
5. $x = y^2 \ if \ y \geq 0$ [yes]
6. $x = \ln(y) \ if \ y > 0$ [yes]
7. $x = y^3 \ if \ y \geq 0$ [yes]
8. $x = -\frac{1}{y}$ [yes – but undefined if $y = 0$]
9. $x = \max(y^2, y^3) \ if \ y \geq 0$ [yes]
10. $x = 2y - y^2$ [no]

Property:

If $U_2(.)$ is a monotone transformation of $U_1(.)$, i.e. $U_2(.) = f(U_1(.))$ where $f(.)$ is monotone in U_1 as defined earlier, then:

- – U_1 and U_2 exhibit identical preference rankings
  – MRS of $U_1(\overline{U})$ and $U_2(\overline{U})$
  $\Longrightarrow U_1$ and U_2 are equivalent for consumer theory

Example: $U(x, y) = x^{\alpha} y^{\beta}$ (Cobb-Douglas)



What is the MRS along an indifference curve U_0?

$$
\begin{aligned}
U_0 &= x_0^\alpha y_0^\beta \\
dU_0 &= \alpha x_0^{\alpha-1} y_0^\beta dx + \beta x_0^\alpha y_0^{\beta-1} dy \\
\left.\frac{dy}{dx}\right|_{U=U_0} &= -\frac{\alpha x_0^{\alpha-1} y_0^\beta}{\beta x_0^\alpha y_0^{\beta-1}} = \frac{\alpha}{\beta}\frac{y_0}{x_0} = \frac{\partial U/\partial x}{\partial U/\partial y}
\end{aligned}
$$

Consider now a monotonic transformation of U:

$$
\begin{aligned}
U^1(x,y) &= x^\alpha y^\beta \\
U^2(x,y) &= \ln(U^1(x,y)) \\
U^2 &= \alpha \ln x + \beta \ln y
\end{aligned}
$$

What is the MRS of U^2 along an indifference curve such that $U^2 = \ln U_0$?

$$
\begin{aligned}
U_0^2 &= \ln U_0 = \alpha \ln x_0 + \beta \ln y_0 \\
dU_0^2 &= \frac{\alpha}{x_0} dx + \frac{\beta}{y_0} dy = 0 \\
\left.\frac{dy}{dx}\right|_{U^2=U_0^2} &= \frac{\alpha}{\beta}\frac{y_0}{x_0} = \frac{\partial U/\partial x}{\partial U/\partial y}
\end{aligned}
$$

which is the same as we derived for U^1.

How do we know that monotonic transformations always preserve the MRS of a utility function?

Let $U = f(x, y)$ be a utility function

Let $g(U)$ be a monotonic transformation of $U = f(x, y)$

The MRS of $g(U)$ along an indifference curve where $U_0 = f(x_0, y_0)$ and $g(U_0) = g(f(x_0, y_0))$

By totally differentiating this equality we can obtain the MRS.

$$\begin{aligned} dg(U_0) &= g'(f(x_0, y_0))f_x(x_0, y_0)dx + g'(f(x_0, y_0))f_y(x_0, y_0)dy \\ -\frac{dy}{dx}\bigg|_{g(U)=g(U_0)} &= \frac{g'(f(x_0, y_0))f_x(x_0, y_0)}{g'(f(x_0, y_0))f_y(x_0, y_0)} = \frac{f_x(x_0, y_0)}{f_y(x_0, y_0)} = \frac{\partial U/\partial x}{\partial U/\partial y} \end{aligned}$$

which is the MRS of the original function $U(x, y)$.